Lec 5: 02/01/2012

Special Relativity (Cont'd):

As we mentioned, a tensor of rank 2 is transformed as follows when going from a frame O to another frame O':

$$T'^{\nu\mu} = \frac{\partial x'^{\nu}}{\partial x^{\rho}} \frac{\partial x'^{\mu}}{\partial x^{\sigma}} T^{\rho\sigma}$$

A useful such tensor is that of the electromagnetic field tensor $F^{\nu\mu}$. Recall that the electric field $\vec{E}$ and magnetic field $\vec{B}$ can be expressed in terms of the scalar potential Φ and vector potential $\vec{A}$:

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \quad \text{(Gaussia units used)}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

In passing, we note that $\vec{E}$ and $\vec{B}$ are the physical quantities that can be measured. $\vec{A}$ and Φ are not physical since different values of $\vec{A}$ and Φ can result in the same $\vec{E}$ and $\vec{B}$. This redundancy can be seen by making a transformation (called gauge transformation):

$$\vec{A} \to \vec{A} - \vec{\nabla}\chi \quad , \quad \Phi \to \Phi - \frac{1}{c}\frac{\partial \chi}{\partial t} \quad (\chi \text{ arbitrary function})$$

This will result in the same $\vec{E}$ and $\vec{B}$ as before.

It turns out that Φ and $\vec{A}$ form a four-vector:

$$A^{\nu} = (\Phi, \vec{A})$$

The electromagnetic field tensor $F^{\nu\mu}$ is defined as:

$$F^{\nu\mu} = \frac{\partial A^{\mu}}{\partial x_{\nu}} - \frac{\partial A^{\nu}}{\partial x_{\mu}}$$

It can be written as a 4x4 matrix, which is antisymmetric:

$$F^{\nu\mu} = \begin{bmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -B^z & B^y \\ E^y & B^z & 0 & -B^x \\ E^z & -B^y & B^x & 0 \end{bmatrix}$$

The transformation properties of tensors can be used to find the relation between the electric and magnetic fields in two frames O and O'. Assuming that O' moves with a velocity v (in the x direction) relative to O, we have:

$E'^x = E^x$

$E^{y'} = \gamma(E^y - \beta B^z)$

$E^{z'} = \gamma(E^z + \beta B^y)$

One can also see two useful Lorentz invariant quantities by contracting $F^{\mu\nu}$ with itself as follows:

$F^{\mu\nu} F_{\mu\nu} \propto (|\vec{E}|^2 - |\vec{B}|^2)$

$\varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \propto \vec{E} \cdot \vec{B}$

Here $\varepsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor (the Levi-Civita tensor).

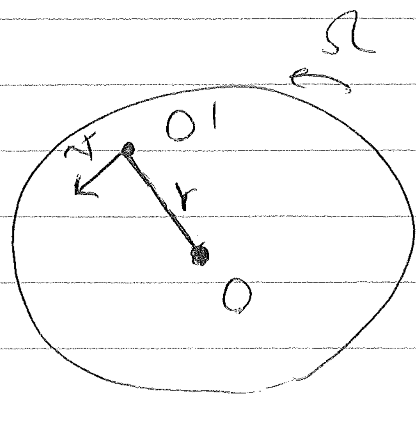
One important consequence is that if $\vec{E} \perp \vec{B}$ in one frame (hence $\vec{E}\cdot\vec{B}=0$), then $\vec{E}\perp\vec{B}$ in all frames.

Also, a purely electric field in one frame cannot look like a purely magnetic field in another frame (since it amounts to a sign change in $|\vec{E}|^2-|\vec{B}|^2$).

Photon Redshift in a Gravitational Potential:

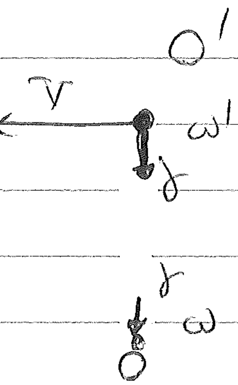
Finally, we derive an expression for the gravitational redshift of photons emitted from the surface of an astrophysical object. A heuristic way to find this is by using the expression of the Doppler shift in special relativity and the equivalence principle.

Lets consider a rotating disk with constant angular velocity Ω:



An observer is in the inertial frame O at the center of the disk. Another inertial frame O' has the same velocity as the instantaneous velocity of a point at a distance r from the center, which is given by $v = \omega r$.

A photon with frequency ω' emitted from this frame will reach the observer at O undergoing transverse Doppler shift:



Recall the expression for the transverse Doppler shift:

$$\omega' = \gamma\omega \Rightarrow \omega = \frac{1}{\gamma}\omega' = \sqrt{1 - \frac{v^2}{c^2}}\,\omega'$$

At a radial distance r from the center, the radial component of the acceleration is $a = -\Omega^2 r$. We can now use to translate this acceleration into a gravitational acceleration.

According to the equivalence principle, an accelerating observer is equivalent to an observer in a gravitational field such that $\vec{g} = -\vec{a}$. This leads to:

$g(r) = -a(r) = \Omega^2 r \Rightarrow U(r) = -\frac{1}{2}\Omega^2 r^2 = -\frac{1}{2}v^2$

The relation between ω and ω' obtained above therefore results in a relation between ω and ω' in a gravitational potential $U(r)$:

$\omega = \sqrt{1 + \frac{2U(r)}{c^2}}\,\omega'$

In the limit of weak gravitational field $U(r) \ll c^2$ (this is equivalent to a slowly rotating this $v^2 \ll c^2$), we have:

$\omega \simeq \left(1 + \frac{U(r)}{c^2}\right)\omega'$

This is exactly what one finds for the photon redshift in the Newtonian approximation of the general relativity.

In an astrophysical setting, ω' is the frequency of a photon emitted from the surface of an object, and ω is the frequency measured by an observer far from the object (at zero gravitational potential). U is the gravitational potential at the surface, which is given by:

$$U = -\frac{GM}{R}$$

Here M and R denote the mass and radius of the object.

Redshift of photons emitted from the sun is:

$$\frac{\omega}{\omega'} \approx 1 - \frac{GM_\odot}{R_\odot c^2} \Rightarrow \frac{\omega' - \omega}{\omega'} \sim O(10^{-6})$$

For a white dwarf $M \sim M_\odot$ and $R \sim 10^4$ km, which results in:

$$\frac{\omega' - \omega}{\omega'} \sim O(10^{-4})$$

For a neutron star $M \sim M_\odot$ and $R \sim 10$ km, which leads to:

$$\frac{\omega' - \omega}{\omega'} \sim O(10^{-1})$$